PRINCIPAL PARTS ON THE PROJECTIVE LINE OVER ARBITRARY RINGS

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ABSTRACT. We develop techniques to split explicitly the sheaf of principal parts $\mathcal{P}^k(\mathcal{O}(n))$ as left and right module on the projective line over an arbitrary ring. We then apply the techniques developed to split the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ for all $n \in \mathbf{Z}$ and $k \geq 1$ as left and right $\mathcal{O}$-module on the projective line over any field of characteristic zero giving a complete description of the principal parts on the projective line.

CONTENTS

1. INTRODUCTION

In previous papers (see [4],[5]) the structure of the principal parts on the projective line was studied using explicit and multilinear algebra techniques, and a partial classification was obtained on the projective line and projective space. In this paper the techniques from [4] are generalized to study the splitting type of the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ for all $k \geq 1$ and all $n \in \mathbf{Z}$ as left and right module on the projective line over any ring. The techniques are powerful enough to enable us to give a complete classification of the splitting type of the principal parts on the projective line over any field of characteristic zero. The main result of the paper is the following:

Theorem 1.1. *Let F be a field of characteristic zero. Consider the principal parts $\mathcal{P}^k(\mathcal{O}(d))$ on the projective line $\mathbf{P}^1_F$. The structure of the principal parts as left module is as follows: If $k \geq 1$ and $n < 0$ or $n \geq k$ there exist an isomorphism*

$$(1.1.1) \qquad \mathcal{P}^k(\mathcal{O}(n))^{left} \cong \oplus^{k+1} \mathcal{O}(n-k)$$

of left $\mathcal{O}$-modules. If $0 \leq n < k$ there exist an isomorphism

$$(1.1.2) \qquad \mathcal{P}^k(\mathcal{O}(n))^{left} \cong \mathcal{O}^{n+1} \oplus \mathcal{O}(-k-1)^{k-n}$$

of left $\mathcal{O}$-modules. The structure of the principal parts as right module is as follows: For all $n \in \mathbf{Z}$ and $k \geq 1$ there exist an isomorphism

$$(1.1.3) \qquad\qquad \mathcal{P}^k(\mathcal{O}(n))^{right} \cong \mathcal{O}(n) \oplus \mathcal{O}(n-k-1)^k$$

of right $\mathcal{O}$-modules.

Proof. See Theorem 5.2, 5.3 and 5.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is hoped that the techniques developed in the paper will shed light on the problem of classifying the structure of the principal parts as left and right module on the projective line over any field.

The paper is organized as follows: In section one we define and prove general properties of the sheaf of principal parts. In section two we calculate the transition matrices of the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ for all $k \geq 1$ and $n \in \mathbf{Z}$ on the projective line over the integers. In section three we develop techniques to split the principal parts as left and right module over any ring. In section four we apply the techniques developed in section three to give a complete classification of the splitting type of the principal parts on the projective line over any field of characteristic zero.

2. Sheaves of principal parts

In this section we define and give general properties of the sheaves of principal parts following [1], chapter 16.

Notation. Let in the following X/S be a separated scheme, and consider the two projection maps $p, q : X \times X \to X$. The diagonal embedding $\Delta : X \to X \times X$ is a closed immersion, and we get an exact sequence of sheaves of $\mathcal{O}_{X \times X}$-modules on $X \times X$

$$(2.0.4) \qquad\qquad 0 \to \mathcal{I}^{k+1} \to \mathcal{O}_{X \times X} \to \mathcal{O}_{\Delta^k} \to 0,$$

where $\mathcal{I}$ is the sheaf of ideals defining the diagonal in $X \times X$.

Definition 2.1. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. We define the *k'th order sheaf of principal parts* of $\mathcal{E}$, to be

$$\mathcal{P}^k_X(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^k} \otimes_{X \times X} q^*\mathcal{E}).$$

We write $\mathcal{P}^k_X$ for the module $\mathcal{P}^k_X(\mathcal{O}_X)$.

When it is clear from the context which scheme we are working on, we write $\mathcal{P}^k(\mathcal{E})$ instead of $\mathcal{P}^k_X(\mathcal{E})$.

Proposition 2.2. *Let X/S be smooth and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. There exists an exact sequence*

$$0 \to S^k(\Omega^1_X) \otimes \mathcal{E} \to \mathcal{P}^k_X(\mathcal{E}) \to \mathcal{P}^{k-1}_X(\mathcal{E}) \to 0$$

of left $\mathcal{O}_X$-modules, where $k = 1, 2, \ldots$.

Proof. See [7], section 4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

From Proposition 2.2 it follows that for a smooth morphism $X \to S$ of relative dimension n, and $\mathcal{E}$ a locally free sheaf on X of rank e, the principal parts sheaf $\mathcal{P}^k(\mathcal{E})$ is locally free of rank $e\binom{n+k}{n}$.

Proposition 2.3. *Let $f : X \to Y$ be a map of smooth schemes over S, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module. There exists a commutative diagram of exact sequences*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S^k(f^*\Omega^1_Y) \otimes f^*\mathcal{E} & \longrightarrow & f^*\mathcal{P}^k_Y(\mathcal{E}) & \longrightarrow & f^*\mathcal{P}^{k-1}_Y(\mathcal{E}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^k(\Omega^1_X) \otimes f^*\mathcal{E} & \longrightarrow & \mathcal{P}^k_X(f^*\mathcal{E}) & \longrightarrow & \mathcal{P}^{k-1}_X(f^*\mathcal{E}) & \longrightarrow & 0
\end{array}
$$

of left $\mathcal{O}_X$-modules for all $k = 1, 2, \ldots$.

Proof. See [8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Note that it follows from Proposition 2.3 that for any open set $U \subseteq X$ of a scheme X smooth over S there is an isomorphism

$$(2.3.1) \qquad\qquad\qquad\qquad \mathcal{P}^k_X(\mathcal{E})|_U \cong \mathcal{P}^k_U(\mathcal{E}|_U)$$

of left $\mathcal{O}_U$-modules: Let $i : U \to X$ be the inclusion map. It follows that i is smooth. By induction and applying the 5-lemma Proposition 2.3 proves the isomorphism 2.3.1. Hence the sheaf of principal parts can be studied using local calculations.

Given any $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ one may define the *sheaf of differential operators* of order k from $\mathcal{F}$ to $\mathcal{G}$ following [1] section 16.8, denoted $\mathcal{D}^k_X(\mathcal{F}, \mathcal{G})$, and there exists an isomorphism

$$(2.3.2) \qquad\qquad\qquad\qquad Hom_X(\mathcal{P}^k_X(\mathcal{F}), \mathcal{G}) \cong \mathcal{D}^k_X(\mathcal{F}, \mathcal{G})$$

of sheaves of abelian groups. The sheaf of abelian groups $\mathcal{D}^k_X(\mathcal{F}, \mathcal{G})$ is naturally a sheaf of $\mathcal{O}_X$-bimodules and it follows from the isomorphism (2.3.2) that the sheaf of principal parts $\mathcal{P}^k_X(\mathcal{F})$ is also a sheaf of $\mathcal{O}_X$-bimodules. This means that for any open set U of X, and local sections $a, b \in \mathcal{O}(U)$ there exists a left and right multiplication on the module $\mathcal{P}^k(\mathcal{E})(U)$ with the property that for all w in $\mathcal{P}^k(\mathcal{E})(U)$ we have

$$a(wb) = (aw)b.$$

We write $\mathcal{P}^k(\mathcal{E})^{left}$ (resp. $\mathcal{P}^k(\mathcal{E})^{right}$) to specify we are considering the left (resp. right) structure.

Note that for X smooth over S and $\mathcal{E}$ locally free of finite rank, it follows that $\mathcal{P}^k_X(\mathcal{E})$ is locally free of finite rank as left and right $\mathcal{O}_X$-module separately.

3. Calculation of transition matrices over $\mathbf{Z}$

In this section we calculate the transition-matrices L^k_n and R^k_n of the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ on the projective line over $\mathbf{Z}$ for all $n \in \mathbf{Z}$ and all $k \geq 1$. The structure-matrices defining the principal parts on the projective line are matrices with entries being Laurent-polynomials with binomial coefficients, hence the principal parts are naturally defined over $\mathbf{Z}$. The matrices L^k_n and R^k_n are elements of the group $\mathrm{GL}(k + 1, \mathbf{Z}[t, t^{-1}])$. The matrix L^k_n define the left $\mathcal{O}$-module structure, and the matrix R^k_n define the right $\mathcal{O}$-module structure. Note that in this section all tensor products are over $\mathbf{Z}$.

Notation. Consider the projective line $\mathbf{P} = \mathbf{P}^1_{\mathbf{Z}}$ over the integers $\mathbf{Z}$. By definition $\mathbf{P}$ equals $\mathrm{Proj}\,\mathbf{Z}[x_0, x_1]$. Let $U_i = \mathrm{Spec}\,\mathbf{Z}[x_0, x_1]_{(x_i)}$. Let furthermore $t = \frac{x_1}{x_0} \otimes 1$ and $u = 1 \otimes \frac{x_1}{x_0}$ under the isomorphism

$$\mathbf{Z}[\frac{x_1}{x_0}] \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{x_1}{x_0}] \cong \mathbf{Z}[u, t].$$

It follows from Definition 2.1 and Proposition 2.3.1 that the sheaf of principal parts $\mathcal{P}^k(\mathcal{O}(n))|_{U_0}$ as a left $\mathcal{O}_{U_0}$-module equals

$$\mathbf{Z}[t]\{1 \otimes x_0^n, dt \otimes x_0^n, \ldots, dt^k \otimes x_0^n\}$$

as a free $\mathbf{Z}[t]$-module where $dt^p = (u - t)^p$. Let $\frac{1}{t} = \frac{x_0}{x_1} \otimes 1$ and $\frac{1}{u} = 1 \otimes \frac{x_0}{x_1}$ under the isomorphism

$$\mathbf{Z}[\frac{x_0}{x_1}] \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{x_0}{x_1}] \cong \mathbf{Z}[\frac{1}{u}, \frac{1}{t}].$$

It follows from Definition 2.1 and Proposition 2.3.1 that $\mathcal{P}^k(\mathcal{O}(n))|_{U_1}$ as a left $\mathcal{O}_{U_1}$-module equals

$$\mathbf{Z}[s]\{1 \otimes x_1^n, ds \otimes x_1^n, \ldots, ds^k \otimes x_1^n\}$$

as a free $\mathbf{Z}[s]$-module, where $s = \frac{1}{t}$ and $ds^p = (\frac{1}{u} - \frac{1}{t})^p$. Let

$$C = \{1 \otimes x_0^n, dt \otimes x_0^n, \ldots, dt^k \otimes x_0^n\}$$

and

$$C' = \{1 \otimes x_1^n, ds \otimes x_1^n, \ldots, ds^k \otimes x_1^n\}.$$

We aim to describe the principal parts by calculating the transition-matrix

$$L_n^k = [I]_C^{C'}$$

between the bases C' and C as an element of the group $\mathrm{GL}(k+1, \mathbf{Z}[t, t^{-1}])$. The following theorem gives the structure-matrix L_n^k describing the left $\mathcal{O}$-module structure of the principal parts on the projective line over $\mathbf{Z}$.

Theorem 3.1. *Consider $\mathcal{P}^k(\mathcal{O}(n))$ as left $\mathcal{O}$-module on $\mathbf{P}$. In the case $n < 0$ the structure-matrix L_n^k is given by the following formula:*

$$(3.1.1) \qquad ds^p \otimes x_1^n = \sum_{j=p}^{k} (-1)^j \binom{j - n - 1}{p - n - 1} t^{n-p-j} dt^j \otimes x_0^n,$$

where $0 \le p \le k$. If $n \ge 0$ the matrix L_n^k is given as follows:
    Case $1 \le k \le n$:

$$(3.1.2) \qquad ds^p \otimes x_1^n = \sum_{j=p}^{k} (-1)^p \binom{n - p}{j - p} t^{n-p-j} dt^j \otimes x_0^n,$$

where $0 \le p \le k$.
    Case $n < k$:

$$(3.1.3) \qquad ds^p \otimes x_1^n = \sum_{j=p}^{n} (-1)^p \binom{n - p}{j - p} t^{n-p-j} dt^j \otimes x_0^n,$$

if $1 \le p \le n$, and

$$(3.1.4) \qquad ds^p \otimes x_1^n = \sum_{j=p}^{k} (-1)^j \binom{j - n - 1}{p - n - 1} t^{n-p-j} dt^j \otimes x_0^n,$$

if $n < p \le k$.

Proof. Case 3.1.1: let $0 \leq p \leq k$.

$$ds^p \otimes x_1^n = (\frac{1}{u} - \frac{1}{t})^p u^n \otimes x_0^n = (-1)^p \frac{1}{t^p} dt^p (t + dt)^{n-p} \otimes x_0^n =$$

$$(-1)^p \frac{1}{t^p} dt^p \frac{1}{(t + dt)^{p-n}} \otimes x_0^n = (-1)^p \frac{1}{t^p} dt^p \frac{1}{t^{p-n}(1 + dt/t)^{p-n}} \otimes x_0^n =$$

We have the formula

$$\frac{1}{(1 + x)^l} = \sum_{i \geq 0} (-1)^i \binom{i + l - 1}{l - 1} x^i$$

and if we let $x = dt/t$ and $l = p - n$ we get

$$(-1)^p \frac{1}{t^{2p-n}} dt^p \sum_{i \geq 0} (-1)^i \binom{i + p - n - 1}{p - n - 1} \frac{1}{t^i} dt^i \otimes x_0^n =$$

$$\sum_{i=0}^{k-p} (-1)^{i+p} \binom{i + p - n - 1}{p - n - 1} t^{n-2p-i} dt^{i+p} \otimes x_0^n.$$

Changing index with $j = i + p$ we get

$$\sum_{j=p}^{k} (-1)^j \binom{j - n - 1}{p - n - 1} t^{n-p-j} dt^j \otimes x_0^n,$$

and equation 3.1.1 is proved. We leave the proofs of equations 3.1.2-3.1.4 as an excercise.

$\square$

We use similar calculations as in the Theorem 3.1 to study the structure-matrix R_n^k defining the principal parts as right $\mathcal{O}$-module on the projective line over $\mathbf{Z}$.

Notation. It follows from Definition 2.1 and Proposition 2.3.1 that the principal parts $\mathcal{P}^k(\mathcal{O}(n))|_{U_0}$ as a right $\mathcal{O}_{U_0}$-module equals

$$\mathbf{Z}[u]\{1 \otimes x_0^n, du \otimes x_0^n, \ldots, du^k \otimes x_0^n\}$$

as a free $\mathbf{Z}[u]$-module, where $du^p = (t - u)^p$. It follows that $\mathcal{P}^k(\mathcal{O}(n))|_{U_1}$ as a right $\mathcal{O}_{U_1}$-module equals

$$\mathbf{Z}[v]\{1 \otimes x_1^n, dv \otimes x_1^n, \ldots, dv^k \otimes x_1^n\}$$

as a free $\mathbf{Z}[v]$-module, where $v = \frac{1}{u}$ and $dv^p = (\frac{1}{t} - \frac{1}{u})^p$. Let

$$D = \{1 \otimes x_0^n, du \otimes x_0^n, \ldots, du^k \otimes x_0^n\}$$

and

$$D' = \{1 \otimes x_1^n, dv \otimes x_1^n, \ldots, dv^k \otimes x_1^n\}.$$

We wish to calculate the transition-matrix

$$R_n^k = [I]_D^{D'}$$

describing the right module structure on the principal parts. The following theorem gives the structure-matrix R_n^k:

Theorem 3.2. *Consider $\mathcal{P}^k(\mathcal{O}(n))$ as right $\mathcal{O}$-module on $\mathbf{P}$. If $n \in \mathbf{Z}$ and $k \geq 1$ the following formulas hold:*

$$(3.2.1) \qquad\qquad\qquad 1 \otimes x_1^n = u^n \otimes x_0^n$$

and

$$(3.2.2) \qquad\qquad dv^p \otimes x_1^n = \sum_{j=p}^{k} (-1)^j \binom{j-1}{p-1} u^{n-p-j} du^j \otimes x_0^n,$$

where $1 \leq p \leq k$.

Proof. Equation 4.2.1 is trivial hence we prove equation 4.2.2: Let $1 \leq p \leq k$. We get

$$dv^p \otimes x_1^n = (\frac{1}{t} - \frac{1}{u})^p u^n \otimes x_0^n = (-1)^p u^{n-p} du^p \frac{1}{(u+du)^p} \otimes x_0^n =$$

$$(-1)^p u^{n-2p} du^p \frac{1}{(1 + \frac{1}{u}du)^p} \otimes x_0^n.$$

Since we have the formula

$$\frac{1}{(1+x)^l} = \sum_{i \geq 0} (-1)^i \binom{i+l-1}{l-1} x^i$$

We get

$$(-1)^p u^{n-2p} du^p \sum_{i=0}^{k} (-1)^i \binom{i+p-1}{p-1} \frac{1}{u^i} du^i \otimes x_0^n.$$

Multiplying we get

$$\sum_{i=0}^{k-p} (-1)^{i+p} \binom{i+p-1}{p-1} u^{n-2p-i} du^{i+p} \otimes x_0^n.$$

Letting $j = i + p$ we get

$$\sum_{j=p}^{k} (-1)^j \binom{j-1}{p-1} u^{n-p-j} du^j \otimes x_0^n,$$

and equation 4.2.2 is proved, and the theorem follows. $\square$

The Theorems 3.1 and 3.2 describe completely the matrices L_n^k and R_n^k for all $n \in \mathbf{Z}$ and $k \geq 1$. By [2] Theorem 2.1 and [3] Theorem 3.1 we know that any finite rank locally free sheaf on $\mathbf{P}^1$ over any field splits into a direct sum of invertible sheaves. The formation of principal parts commutes with direct sums, hence it follows from Theorems 3.1 and 3.2 that we have calculated the transition matrix of $\mathcal{P}^k(\mathcal{E})$ for any locally free finite rank sheaf $\mathcal{E}$ as left and right $\mathcal{O}$-module on the projective line over any field.

4. Maps of sheaves and linear equations

In this section we develop criteria to split the principal parts on the projective line over arbitrary rings. We prove existence of systems of linear equations with the property that solutions to these systems with coefficients in a ring A gives rise to a splitting of the principal parts on $\mathbf{P}^1_A$ - the projective line over A. Let $\mathbf{P}^1_A = \mathrm{Proj}(A[x_0, x_1])$ and let $t = x_1/x_0$ and $s = 1/t$. It follows that

$$\mathcal{P}^k(\mathcal{O}(n))|_{U_0} = A[t]\{1 \otimes x_0^n, \cdots, dt^k \otimes x_0^n\}$$

and

$$\mathcal{P}^k(\mathcal{O}(n))|_{U_1} = A[s]\{1 \otimes x_1^n, \cdots, ds^k \otimes x_1^n\},$$

where $U_i \subseteq \mathbf{P}^1_A$ with $i = 0, 1$ are the basic open subsete of the projective line over A. Let in the following $n < 0$, $k \geq 1$ and recall the formula from Theorem 3.1(with $0 \leq p \leq k$):

$$(4.0.3) \qquad ds^p \otimes x_1^n = \sum_{j=p}^{k} (-1)^j \binom{j-n-1}{p-n-1} t^{n-p-j} dt^j \otimes x_0^n.$$

Let furthermore for $0 \leq j \leq i \leq k$ x_j^i be independent variables over $\mathbf{Z}$ and let $0 \leq l \leq k$. Let also $J = \mathcal{P}^k(\mathcal{O}(d))$ Make the following definition:

$$\phi_l^1(x_1^{n-k}) = \sum_{p=0}^{l} x_{l-p}^l t^{p-l} ds^p \otimes x_1^n.$$

We seek to define for each $l = 0, .., k$ a map of left modules

$$\phi_l : \mathcal{O}(n-k) \to J$$

and we define

$$\phi_l^0(x_0^{n-k}) = t^{k-n} x_1^{n-k} =$$

$$t^{k-n} \sum_{p=0}^{l} x_{l-p}^l t^{p-l} ds^p \otimes x_1^n = \sum_{p=0}^{l} x_{l-p}^l t^{k-n-l+p} ds^p \otimes x_1^n.$$

We seek to give relations between the variables x_j^i in order for the maps ϕ_l^0 and ϕ_l^1 to glue to a well-defined map of sheaves. We get using formula 4.0.3 the following:

$$\phi_k^0(x_0^{n-k}) = \sum_{p=0}^{l} x_{l-p}^l t^{k-n-l+p} \sum_{j=p}^{k} (-1)^j \binom{j-n-1}{p-n-1} t^{n-p-j} dt^j \otimes x_0^n =$$

$$\sum_{p=0}^{l} \sum_{j=p}^{k} (-1)^j x_{l-p}^l \binom{j-n-1}{p-n-1} t^{k-l-j} dt^j \otimes x_0^n.$$

By convention

$$\binom{j-n-1}{p-n-1} = 0$$

when $j < p$ hence we get the expression

$$\sum_{p=0}^{l} \sum_{j=0}^{k} (-1)^j x_{l-p}^l \binom{j-n-1}{p-n-1} t^{k-l-j} dt^j \otimes x_0^n =$$

$$\sum_{j=0}^{k} \sum_{p=0}^{l} (-1)^j x_{l-p}^l \binom{j-n-1}{p-n-1} t^{k-l-j} dt^j \otimes x_0^n.$$

$$\sum_{j=0}^{k}(-1)^j(\sum_{p=0}^{l} x_{l-p}^l \binom{j-n-1}{p-n-1})t^{k-l-j}dt^j \otimes x_0^n.$$

Define

$$c_j^l = \sum_{p=0}^{l} x_{l-p}^l \binom{j-n-1}{p-n-1}.$$

We get the expression

$$\phi_l^0(x_0^{n-k}) = \sum_{j=0}^{k}(-1)^j c_j^l t^{k-l-j}dt^j \otimes x_0^n =$$

$$\sum_{j=0}^{k-l-1}(-1)^j c_j^l t^{k-l-j}dt^j \otimes x_0^n + (-1)^{k-l}c_{k-l}^l dt^{k-l} \otimes x_0^n +$$

$$(-1)^{k-l+1}c_{k-l+1}^l \frac{1}{t}dt^{k-l+1} \otimes x_0^n + \cdots + (-1)^k c_k^l \frac{1}{t^k}dt^k \otimes x_0^n.$$

We see that if the system of linear equations

(4.0.4) $$S_{k,n}^l : c_{k-l}^l = 1, c_{k-l+1}^l = 0, ..., c_k^l = 0$$

holds for all $l = 0, .., k$, we get a well defined map of left modules

$$\phi_l : \mathcal{O}(n-k) \to J$$

defined locally as follows:

(4.0.5) $$\phi_l^0(x_0^{n-k}) = \sum_{j=0}^{k-l-1}(-1)^j c_j^l t^{k-l-j}dt^j \otimes x_0^n$$

and

(4.0.6) $$\phi_l^1(x_1^{n-k}) = \sum_{p=0}^{l} x_{l-p}^l t^{p-l}ds^p \otimes x_1^n.$$

Note that the system $S_{k,n}^l$ is defined over $\mathbf{Z}$.

Theorem 4.1. *Let $n < 0$ and $k \geq 1$. If for all $l = 0, .., k$ the systems $S_{k,n}^l$ from 4.0.4 has solutions x_j^i in a ring A with the property that*

(4.1.1) $$x_0^0 x_0^1 \cdots x_0^k \in A^*$$

then there exists an isomorphism

$$\phi : \oplus^{k+1}\mathcal{O}(n-k) \to \mathcal{P}^k(\mathcal{O}(n))^{left}$$

as left modules on $\mathbf{P}_A^1$.

Proof. Assume the system $S_{k,n}^l$ from 4.0.4 has solutions in A for all $l = 0, .., k$. By construction we get a map of left modules $\phi = \oplus_{l\geq 0}\phi_l$

$$\phi : \oplus^{k+1}\mathcal{O}(n-k) \to \mathcal{P}^k(\mathcal{O}(n)).$$

We want to prove that the map ϕ is an isomorphism of left modules. We first consider the matrix of $\phi^0 = \phi|_{U_0}$:

$$\phi^0 = \begin{pmatrix} \pm c_0^0 t^k & \pm c_k^1 t^{k-1} & 0 & 0 & \pm c_0^k \\ \pm c_1^0 t^{k-1} & \pm c_1^1 t^{k-2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \pm c_{k-1}^1 & \vdots & \cdots & \vdots \\ \pm c_k^0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that by definition $|\phi^0| = 1$ which is a unit in A, hence the matrix defines an isomorphism. Consider the matrix $\phi^1 = \phi|_{U_1}$:

$$\phi^1 = \begin{pmatrix} x_0^0 & x_1^1 \frac{1}{t} & \cdots & x_k^k \frac{1}{t^k} \\ 0 & x_0^1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & x_0^k \end{pmatrix}.$$

The determinant $|\phi^1| = x_0^0 x_0^1 \cdots x_0^k \in A^*$ hence it is a unit by definition, hence an isomorphism, and the claim is proved. □

We next consider the structure of $J = \mathcal{P}^k(\mathcal{O}(n))$ in the case when $0 \le n < k$. Let $\mathcal{O}$ have local basis e on U_0 and f on U_1 on $\mathbf{P}_A^1$ where A is an arbitrary ring. Let furthermore y_j^i be independent variables over $\mathbf{Z}$ for all $0 \le j \le i \le n$. Using similar calculations as above we may do the following: Define for $0 \le l \le n$

$$d_j^l = \sum_{p=0}^l (-1)^p y_{l-p}^l \binom{n-p}{j-p}.$$

Let furthermore

$$\phi_l^0(e) = \sum_{j=0}^{n-l} d_j^l t^{n-l-j} dt^j \otimes x_0^n,$$

and

$$\phi_l^1(f) = \sum_{p=0}^l y_{l-p}^l t^{p-l} ds^p \otimes x_1^n.$$

If the system of equations

(4.1.2) $$S_{k,n}^l : d_{n-l}^l = 1, d_{n-l+1}^l = \cdots = d_n^l = 0$$

is satisfied, we get for each $l = 0, .., n$ a map of left modules

$$\phi_l : \mathcal{O} \to J,$$

Let furthermore z_j^i be independent variables over $\mathbf{Z}$ for $0 \le j \le i \le k - n - 1$ and define for $0 \le m \le k - n - 1$

$$e_j^m = \sum_{j=0}^m z_{m-i}^m \binom{j-n-1}{i}.$$

We seek to define maps

$$\psi_m : \mathcal{O}(-k-1) \to J$$

of left modules. Make the following definitions:

$$\psi_m^0(x_0^{-k-1}) = \sum_{j=n+1}^{k-m} (-1)^j e_j^m t^{k-m-j} dt^j \otimes x_0^n$$

and

$$\psi_m^1(x_1^{-k-1}) = \sum_{i=0}^{m} z_{m-i}^m t^{i-m} ds^{n+1+i} \otimes x_1^n.$$

Consider the system of linear equations

(4.1.3) $$T_{k,n}^m : c_{k-m}^m = 1, c_{k-m+1}^m = \cdots = c_k^m = 0.$$

Theorem 4.2. *Let $0 \le n < k$. If the systems $S_{k,n}^l$ and $T_{k,n}^m$ from Equations 4.1.2 and 4.1.3 have solutions in A with the property that*

$$y_0^0 y_0^1 \cdots y_0^n \in A^*$$

and

$$z_0^0 z_0^1 \cdots z_0^{k-(n+1)} \in A^*$$

there exists an isomorphism

$$\psi : \mathcal{O}^{n+1} \oplus \mathcal{O}(-k-1)^{k-n} \cong \mathcal{P}^k(\mathcal{O}(n))^{left}$$

as left modules on $\mathbf{P}_A^1$.

Proof. The proof is similar to the proof of Theorem 4.1 and we leave the details to the reader. $\square$

We next consider the principal parts as right module. Let $n \in \mathbf{Z}$ and $k \ge 1$. Let $\mathbf{P}_A^1 = \mathrm{Proj}(A[x_0, x_1])$ and pick coordinates $u = \frac{x_1}{x_0}$ on U_0 and $v = 1/u$ on U_1. We have isomorphisms

$$\mathcal{P}^k(\mathcal{O}(n))|_{U_0} \cong A[u]\{1 \otimes x_0^n, \cdots, du^k \otimes x_0^n\}$$

and

$$\mathcal{P}^k(\mathcal{O}(n))|_{U_1} \cong A[v]\{1 \otimes x_1^n, \cdots, dv^k \otimes x_1^n\}$$

of free modules. Recall from Theorem 3.2 the formulas

(4.2.1) $$1 \otimes x_1^n = u^n \otimes x_0^n$$

and

(4.2.2) $$dv^p \otimes x_1^n = \sum_{j=p}^{k} (-1)^j \binom{j-1}{p-1} u^{n-p-j} du^j \otimes x_0^n,$$

where $1 \le p \le k$. Define a map of right modules

$$\mathcal{O}(n) \to \mathcal{P}^k(\mathcal{O}(n))^{right}$$

as follows:

$$x_0^n \to 1 \otimes x_0^n$$

and

$$x_1^n \to 1 \otimes x_1^n.$$

It follows from Equation 4.2.1 that we get a well-defined map of right modules. We seek to define for $0 \le l \le k-1$ maps

$$\psi_l : \mathcal{O}(n-k-1) \to \mathcal{P}^k(\mathcal{O}(n))^{right}$$

of right modules. Let for $0 \leq j \leq i \leq k-1$ the variables w_j^i be independent over A. Make the folloting definition:

$$\psi_l^1(x_1^{n-k-1}) = \sum_{i=0}^{l} w_{l-i}^k u^{i-l} dv^{i+1} \otimes x_1^n.$$

We define

$$\psi_l^0(x_0^{n-k-1}) = x_0^{n-k-1} = u^{k+1-n} x_1^{n-k-1} =$$

$$u^{k+1-n} \sum_{i=0}^{l} w_{l-i}^l u^{i-l} dv^{i+1} \otimes x_1^n.$$

Using Equation 4.2.2 we get

$$\psi_l^0(x_0^{n-k-1}) = u^{k+1-n} \sum_{i=0}^{l} w_{l-i}^l u^{i-l} \sum_{j=i+1}^{k} (-1)^j \binom{j-1}{i} u^{n-i-1-j} du^j \otimes x_0^n =$$

$$\sum_{i=0}^{l} \sum_{j=i+1}^{k} (-1)^j w_{l-i}^l \binom{j-1}{i} u^{k-l-j} du^j \otimes x_0^n.$$

We let $\binom{a}{b} = 0$ if $a < b$ and get the expression

$$\sum_{j=1}^{k} (-1)^j \sum_{i=0}^{l} w_{l-i}^l \binom{j-1}{i} u^{k-l-j} du^j \otimes x_0^n.$$

Define

$$f_j^l = \sum_{i=0}^{l} w_{l-i}^l \binom{j-1}{i}.$$

We get

$$\psi_l^0(x_0^{n-k-1}) = \sum_{j=1}^{k} (-1)^j f_j^l u^{k-l-j} du^j \otimes x_0^n =$$

$$\sum_{j=1}^{k-l-1} (-1)^j f_j^l u^{k-l-j} du^j \otimes x_0^n + (-1)^{k-l} f_{k-l}^l du^{k-l} \otimes x_0^n +$$

$$(-1)^{k-l+1} f_{k-l+1}^l 1/u du^{k-l+1} \otimes x_0^n + \cdots + (-1)^k f_k^l 1/u^k du^k \otimes x_0^n.$$

Let $0 \leq l \leq k-1$ and consider the system of linear equations

(4.2.3) $$U_{k,n}^l : f_{k-l}^l = 1, f_{k-l+1}^l = \cdots = f_k^l = 0.$$

If the system $U_{k,n}^l$ from Equation 4.2.3 has a solution in A for all $0 \leq l \leq k-1$ we get by definition well-defined maps

$$\psi_l : \mathcal{O}(n-k-1) \rightarrow \mathcal{P}^k(\mathcal{O}(n))$$

of right modules on $\mathbf{P}_A^1$ hence we get a map

(4.2.4) $$\psi : \mathcal{O}(n) \oplus \mathcal{O}(n-k-1)^k \rightarrow \mathcal{P}^k(\mathcal{O}(n))$$

of right modules.

Theorem 4.3. *Let $n \in \mathbf{Z}$, $k \geq 1$ and $0 \leq l \leq k-1$. If the system $U^l_{k,n}$ from Equation 4.2.3 has a solution with coefficients in A satisfying*

$$w_0^0 w_0^1 \cdots w_0^{k-1} \in A^*$$

there is an isomorphism

$$\psi : \mathcal{O}(n) \oplus \mathcal{O}(n-k-1)^k \to \mathcal{P}^k(\mathcal{O}(n))^{right}$$

of right modules on $\mathbf{P}^1_A$.

Proof. We prove that the map ψ introduced above (see 4.2.4) is an isomorphism. We first consider the map $\psi|_{U_0} = \psi^0$:

$$\psi^0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \pm f_1^0 u^{k-1} & \pm f_1^1 u^{k-2} & \cdots & \pm f_1^{k-1} \\ 0 & \pm f_2^0 u^{k-2} & \pm f_2^1 u^{k-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \pm f_{k-1}^0 u & \pm f_{k-1}^1 & \cdots & 0 \\ 0 & \pm f_k^0 & 0 & \cdots & 0 \end{pmatrix}.$$

It follows that $|\psi^0| = \pm f_k^0 f_{k-1}^1 \cdots f_1^{k-1} = 1$ hence $\psi|_{U_0}$ is an isomorphism. We next consider the map $\psi|_{U_1} = \psi^1$:

$$\psi^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & w_0^0 & w_1^1 1/u & \cdots & \cdots & w_{k-1}^{k-1} \frac{1}{u^{k-1}} \\ 0 & 0 & w_0^1 & \cdots & \cdots & w_{k-2}^{k-1} \frac{1}{u^{k-2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & w_0^{k-1} \end{pmatrix}.$$

It follows that

$$|\psi^1| = w_0^0 w_0^1 \cdots w_0^{k-1} \in A^*$$

hence the map is an isomorphism, and the claim follows. $\square$

The results of this section generalize the results obtained in [4], where only the principal parts as left module is considered.

5. Application: Splitting over fields of characteristic zero

In this section we look at the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ as left and right $\mathcal{O}$-module on the projective line for all $n \in \mathbf{Z}$ and $k \geq 1$ over any field F of characteristic zero. We use the results of section 2 and techniques developed in section 3 to give the splitting-type of the principal parts as left and right module over any field of characteristic zero, settling completely the problem of describing the structure of the principal parts on the projective line. We first consider the splitting-type of the principal parts as left module.

First a technical lemma. Consider the following system of linear equations $A_l \mathbf{x_l} = b_l$:

$$
\begin{pmatrix}
\binom{k+m-l-1}{m-1} & \binom{k+m-l-1}{m} & \binom{k+m-l-1}{m+1} & \cdots & \binom{k+m-l-1}{m+l-1} \\
\binom{k+m-l}{m-1} & \binom{k+m-l}{m} & \binom{k+m-l}{m+1} & \cdots & \binom{k+m-l}{m+l-1} \\
\vdots & \vdots & \vdots & \vdots & \\
\binom{k+m-2}{m-1} & \binom{k+m-2}{m} & \binom{k+m-2}{m+1} & \cdots & \binom{k+m-2}{m+l-1} \\
\binom{k+m-1}{m-1} & \binom{k+m-1}{m} & \binom{k+m-1}{m+1} & \cdots & \binom{k+m-1}{m+l-1}
\end{pmatrix}
\begin{pmatrix}
x_l^l \\ x_{l-1}^l \\ \vdots \\ x_1^l \\ x_0^l
\end{pmatrix}
=
\begin{pmatrix}
1 \\ 0 \\ \vdots \\ 0 \\ 0
\end{pmatrix},
$$

with $l = 0, \ldots k$.

Lemma 5.1. *If the vector* $\mathbf{x_l}$ *is a solution to the system defined above, it follows that* $x_0^l \neq 0$.

Proof. Assume on the contrary that $x_0^l = 0$. We get the following system of linear equations:

$$
\begin{pmatrix}
\binom{k+n-l}{n-1} & \binom{k+n-l}{n} & \binom{k+n-l}{n+1} & \cdots & \binom{k+n-l}{n-1+l-1} \\
\vdots & \vdots & \vdots & \vdots & \\
\binom{k+n-2}{n-1} & \binom{k+n-2}{n} & \binom{k+n-2}{n+1} & \cdots & \binom{k+n-2}{n-1+l-1} \\
\binom{k+n-1}{n-1} & \binom{k+n-1}{n} & \binom{k+n-1}{n+1} & \cdots & \binom{k+n-1}{n-1+l-1}
\end{pmatrix}
\begin{pmatrix}
x_l^l \\ x_{l-1}^l \\ \vdots \\ x_1^l
\end{pmatrix}
=
\begin{pmatrix}
0 \\ 0 \\ \vdots \\ 0
\end{pmatrix},
$$

$$\mathbf{Ax = b}.$$

Using Lemma 6.1 with $a = k + n - l, b = n - 1$ and $l - 1$ we get

$$
|A| = \frac{\prod_{i=0}^{l-1} \binom{k+n-l+i}{n-1+i}}{\prod_{i=0}^{l} \binom{k-l+1+i}{i}} \neq 0
$$

and the lemma is proved. $\qquad\square$

Theorem 5.2. *Consider the principal parts* $\mathcal{P}^k(\mathcal{O}(n))$ *as left module on* $\mathbf{P}_F^1$ *with* $n < 0$ *and* $k \geq 1$ *or* $1 \leq k \leq n$. *There exists an isomorphism*

$$\phi : \mathcal{O}(n-k)^{k+1} \to \mathcal{P}^k(\mathcal{O}(n))^{left}$$

of left $\mathcal{O}_{\mathbf{P}_F^1}$*-modules.*

Proof. The case where $1 \leq k \leq n$ is proved in [4] Proposition 6.3. Consider the case where $n < 0$ and $k \geq 1$. Let $0 \leq l \leq k$. According to Theorem 4.1, there exists for all $0 \leq l \leq k$ systems $S_{k,n}^l$ (see 4.0.4) of linear equations with the property that solutions in F for all $0 \leq l \leq k$ gives rise to a map

$$\phi : \mathcal{O}(n-k)^{k+1} \to \mathcal{P}^k(\mathcal{O}(n))$$

of left $\mathcal{O}_{\mathbf{P}_F^1}$-modules. We seek solutions in F to the following system of equations $S_{k,n}^l : A_l \mathbf{x_l} = b_l$:

$$
\begin{pmatrix}
\binom{k+m-l-1}{m-1} & \binom{k+m-l-1}{m} & \binom{k+m-l-1}{m+1} & \cdots & \binom{k+m-l-1}{m+l-1} \\
\binom{k+m-l}{m-1} & \binom{k+m-l}{m} & \binom{k+m-l}{m+1} & \cdots & \binom{k+m-l}{m+l-1} \\
\vdots & \vdots & \vdots & \vdots & \\
\binom{k+m-2}{m-1} & \binom{k+m-2}{m} & \binom{k+m-2}{m+1} & \cdots & \binom{k+m-2}{m+l-1} \\
\binom{k+m-1}{m-1} & \binom{k+m-1}{m} & \binom{k+m-1}{m+1} & \cdots & \binom{k+m-1}{m+l-1}
\end{pmatrix}
\begin{pmatrix}
x_l^l \\ x_{l-1}^l \\ \vdots \\ x_1^l \\ x_0^l
\end{pmatrix}
=
\begin{pmatrix}
1 \\ 0 \\ \vdots \\ 0 \\ 0
\end{pmatrix},
$$

for all $l = 0, \ldots k$. By Lemma 6.1 the determinant of A_l is

$$|A_l^1| = \frac{\binom{k-l+m-1}{m-1}\binom{k-l+m}{m}\cdots\binom{k+m-1}{m-1+l}}{\binom{k-l}{0}\binom{k-l+1}{1}\cdots\binom{k}{l}}$$

which is non-zero for all l since the characteristic of F is zero, hence there exists a map

$$\phi : \mathcal{O}(n-k)^{k+1} \to \mathcal{P}^k(\mathcal{O}(n))$$

of left $\mathcal{O}$-modules. Consider the maps $\phi^i = \phi|_{U_i}$ for $i = 0, 1$. By the proof of Theorem 4.1 the determinant of ϕ^0 equals

$$|\phi^0| = \pm c_k^0 c_{k-1}^1 c_{k-2}^2 \cdots c_0^k$$

which is nonzero by construction, hence ϕ^0 is an isomorphism. By the proof of Theorem 4.1 the determinant $|\phi^1|$ equals

$$|\phi^1| = x_0^0 x_0^1 x_0^2 \cdots x_0^k$$

which is nonzero by Lemma 5.1. It follows that ϕ^1 is an isomorphism, and the theorem follows. □

Theorem 5.3. *Consider the principal parts $\mathcal{P}^k(\mathcal{O}(n))$ as left $\mathcal{O}_{\mathbf{P}_F^1}$-module on $\mathbf{P}_F^1$ with $0 \leq n < k$. There exists an isomorphism*

$$(5.3.1) \qquad \phi : \mathcal{O}^{n+1} \oplus \mathcal{O}(-k-1)^{k-n} \to \mathcal{P}^k(\mathcal{O}(n))^{left}$$

of left $\mathcal{O}_{\mathbf{P}_F^1}$-modules.

Proof. By Theorem 4.2 there exists systems $S_{k,n}^l$ and $T_{k,n}^m$ (see 4.1.2 and 4.1.3) with the property that solutions in F gives rise to a map

$$\phi : \mathcal{O}^{n+1} \oplus \mathcal{O}(-k-1) \to \mathcal{P}^k(\mathcal{O}(n))$$

of left modules. We seek solutions in F to the following systems of linear equations:
$S_{k,n}^l : A_l \mathbf{x_l} = b_l$

$$\begin{pmatrix} \binom{n}{n} & -\binom{n-1}{n-1} & \binom{n-2}{n-2} & \cdots & (-1)^l\binom{n-l}{n-l} \\ \binom{n}{n-1} & -\binom{n-1}{n-2} & \binom{n-2}{n-3} & \cdots & (-1)^l\binom{n-l}{n-l-1} \\ \vdots & \vdots & \vdots & \vdots & \\ \binom{n}{n-l+1} & -\binom{n-1}{n-l} & \binom{n-2}{n-l-1} & \cdots & (-1)^l\binom{n-l}{n-2l+1} \\ \binom{n}{n-l} & -\binom{n-1}{n-l-1} & \binom{n-2}{n-l-2} & \cdots & (-1)^l\binom{n-l}{n-2l} \end{pmatrix} \begin{pmatrix} y_l^l \\ y_{l-1}^l \\ \vdots \\ y_1^l \\ y_0^l \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

and the system $T_{k,n}^m : B_m \mathbf{y_m} = c_m$

$$\begin{pmatrix} \binom{k-n-1}{0} & \binom{k+n-1}{1} & \binom{k-n-1}{2} & \cdots & \binom{k-n-1}{m} \\ \binom{k-n-2}{0} & \binom{k-n-2}{1} & \binom{k-n-2}{2} & \cdots & \binom{k-n-2}{m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{k-n-m}{0} & \binom{k-n-m}{1} & \binom{k-n-m}{2} & \cdots & \binom{k-n-m}{m} \\ \binom{k-n-m-1}{0} & \binom{k-n-m-1}{1} & \binom{k-n-m-1}{2} & \cdots & \binom{k-n-m-1}{m} \end{pmatrix} \begin{pmatrix} z_m^m \\ z_{m-1}^m \\ \vdots \\ z_1^m \\ z_0^m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

where $l = 0, \ldots, n$ and $m = 0, \ldots, k - n - 1$. The determinant of A_l is by Lemma 6.2 equal to $(-1)^{l+1}$, hence the system $A_l \mathbf{x_l} = b_l$ has solutions in any field F of characteristic zero. The determinant of the matrix B_m equals

$$|B_m| = \frac{\binom{k-n-1}{0}\binom{k-n}{1}\cdots\binom{k-n+m-1}{m}}{\binom{k-n-1}{0}\binom{k-n}{1}\cdots\binom{k-n+m-1}{m}} = 1$$

which is different from zero in F. It follows that the system $B_m \mathbf{y_m} = c_m$ always has solutions in F, hence we get a map

$$\phi : \mathcal{O}^{n+1} \oplus \mathcal{O}(-k-1)^{k-n} \to \mathcal{P}^k(\mathcal{O}(n))$$

of left $\mathcal{O}_{\mathbf{P}^1_F}$-modules defined as follows: Following the discussion preceeding Theroem 4.2 and we define maps ϕ^i_l for $l = 0, \ldots, n$ on U_0 as follows:

(5.3.2)
$$\phi^0_l(e) = \sum_{j=0}^{n-l} d^l_j t^{n-l-j} dt^j \otimes x^n_0$$

On U_1 we define:

(5.3.3)
$$\phi^1_l(f) = \sum_{p=0}^{l} y^l_{l-p} t^{p-l} ds^p \otimes x^n_1.$$

We define for $0 \leq m \leq k-n-1$ maps ψ^i_l as follows:

(5.3.4)
$$\psi^0_m(x^{-k-1}_0) = \sum_{j=k+1}^{k-m} (-1)^j e^m_j t^{k-m-j} dt^j \otimes x^n_0$$

On U_1:

(5.3.5)
$$\psi^1_l(x^{-k-1}_1) = \sum_{i=0}^{m} z^m_{m-i} t^{i-m} ds^{n+1+i} \otimes x^n_1.$$

Let $\phi = \oplus^n_{i=0} \phi_l \oplus^{k-n-1}_{i=0} \psi_i$. We claim that ϕ is an isomorphism. We consider the map $\phi|_{U_0}$: Its determinant equals

$$|\phi|_{U_0}| = \pm d^0_n d^1_{n-1} \cdots d^m_0 e^0_k e^1_{k-1} \cdots e^{k-n-1}_{n+1}$$

which is different from zero by construction, hence $\phi|_{U_0}$ is an isomorphism. We next consider the map ϕ_{U_1}. Its determinant equals

$$|\phi|_{U_1}| = y^0_0 y^1_0 \cdots y^n_0 z^0_0 z^1_0 z^{k-n-1}_0$$

which is nonzero by an argument similar to the one in Lemma 5.1, hence $\phi|_{U_1}$ is an isomorphism, and the theorem is proved. $\square$

We get a similar theorem on the splitting-type of the principal parts as right module:

Theorem 5.4. *Let F be a field of characteristic zero. Consider $\mathcal{P}^k(\mathcal{O}(n))$ as right $\mathcal{O}$-module on $\mathbf{P}^1_F$ for all $n \in \mathbf{Z}$ and $k \geq 1$. There exists an isomorphism*

$$\psi : \mathcal{O}(n) \oplus \mathcal{O}(n-k-1)^k \to \mathcal{P}^k(\mathcal{O}(n))^{right}$$

of right $\mathcal{O}$-modules.

Proof. Define a map

$$\psi_{-1} : \mathcal{O}(n) \to \mathcal{P}^k(\mathcal{O}(n))^{right}$$

as follows:

$$\psi_{-1}(x^n_0) = 1 \otimes x^n_0$$

and

$$\psi_{-1}(x^n_1) = 1 \otimes x^n_1$$

According to Theorem 4.3 we need to prove existence of solutions of the systems $U^l_{k,n}$ (see 4.2.3) with coefficients in F for all $0 \leq l \leq k-1$.

Existence of the map f is by definition equivalent to the existence of solutions in F of the following systems of linear equations $A_l \mathbf{x_l} = b_l$:

$$
\begin{pmatrix}
\binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{l} \\
\binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \cdots & \binom{k-2}{l} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\binom{k-l}{0} & \binom{k-l}{1} & \binom{k-l}{2} & \cdots & \binom{k-l}{l} \\
\binom{k-l-1}{0} & \binom{k-l-1}{1} & \binom{k-l-1}{2} & \cdots & \binom{k-l-1}{l}
\end{pmatrix}
\begin{pmatrix}
w_l^l \\ w_{l-1}^l \\ \vdots \\ w_1^l \\ w_0^l
\end{pmatrix}
\begin{pmatrix}
1 \\ 0 \\ \vdots \\ 0 \\ 0
\end{pmatrix},
$$

where $l = 0, \ldots, k - 1$. The determinant of the matrix A_l equals

$$
|A_l| = \frac{\binom{k-1}{0}\binom{k}{1}\cdots\binom{k+l-1}{l}}{\binom{k-1}{0}\binom{k}{1}\cdots\binom{k+l-1}{l}} = 1
$$

which is a unit, hence the map ϕ exists over the field F.

According to the discussion preceeding Theorem 4.3 we may define for $0 \leq l \leq k - 1$ maps ψ_l as follows: On U_0 we define

(5.4.1)
$$
\psi_l^0(x_0^{n-k-1}) = \sum_{j=0}^{k-l} (-1)^j f_j^l u^{k-l-j} du^j \otimes x_0^n.
$$

Define a map on U_1:

(5.4.2)
$$
\psi_l^1(x_1^{n-k-1}) = \sum_{i=0}^{l} w_{l-i}^l u^{i-l} dv^{i+1} \otimes x_1^n,
$$

Define $\psi = \oplus_{i=-1}^{k-1} \psi_i$. The determinant of $\phi|_{U_0}$ equals

$$
|\psi|_{U_0}| = \pm f_k^0 f_{k-1}^1 \cdots f_0^k
$$

which is non-zero by construction, hence $\psi|_{U_0}$ is an isomorphism. The determinant of $\psi|_{U_1}$ equals

$$
|\psi|_{U_1}| = w_0^0 w_0^1 \cdots w_0^{k-1}
$$

which is non-zero using an argument similar to the one in Lemma 5.1, hence $\psi|_{U_1}$ is an isomorphism, and the theorem is proved. $\qquad\square$

Note that the determinant of the matrix A_l from the proof of Theorem 4.2.1 is one, hence the map constructed exists over an arbitrary ring.

The Theorems 5.2, 5.3 and 5.4 generalize Proposition 6.3 and Theorem 7.1 in [4], settling completely the problem of describing the principal parts on the projective line over a field of characteristic zero. Since formation of principal parts commute with direct sums and because on the projective line over any field any finite rank locally free sheaf splits into invertible sheaves ([2],[3]), we have described the splitting-type of the principal parts of any locally free finite rank sheaf as left and right module over any field of characteristic zero, generalizing results obtained in [4],[9] and [11]. The problem of classifying principal parts on the projective line over an arbitrary field will be adressed in a future paper.

6. Appendix: calculations of determinants

In this section we calculate some determinants appearing in section 3 and 4.

Lemma 6.1. *Assume a, b are integers with $b \leq a$. Then the following is true for all $l \geq 1$: The determinant of the matrix*

$$(6.1.1) \qquad M = \begin{pmatrix} \binom{a}{b} & \binom{a}{b+1} & \binom{a}{b+2} & \cdots & \binom{a}{b+l} \\ \binom{a+1}{b} & \binom{a+1}{b+1} & \binom{a+1}{b+2} & \cdots & \binom{a+1}{b+l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{a+l-1}{b} & \binom{a+l-1}{b+1} & \binom{a+l-1}{b+2} & \cdots & \binom{a+l-1}{b+l} \\ \binom{a+l}{b} & \binom{a+l}{b+1} & \binom{a+l}{b+2} & \cdots & \binom{a+l}{b+l} \end{pmatrix}$$

equals

$$(6.1.2) \qquad |M| = \frac{\prod_{i=0}^{l} \binom{a+i}{b+i}}{\prod_{i=0}^{l} \binom{a-b+i}{i}}$$

Proof. One easily shows the following formula:

$$(6.1.3) \qquad \binom{a+i}{b+j} - \frac{a+i+1-b-l}{a+i+1} \binom{a+i+1}{b+j} = \frac{l-j}{b+j} \binom{a+i}{b-1+j}.$$

Apply equation 6.1.3 to the matrix 6.1.1 to get the matrix

$$(6.1.4) \qquad \begin{pmatrix} \frac{l}{b}\binom{a}{b-1} & \frac{l-1}{b+1}\binom{a}{b} & \frac{l-2}{b+2}\binom{a}{b+1} & \cdots & \frac{1}{b+l-1}\binom{a}{b+l-2} & 0 \\ \frac{l}{b}\binom{a+1}{b-1} & \frac{l-1}{b+1}\binom{a+1}{b} & \frac{l-2}{b+2}\binom{a+1}{b+1} & \cdots & \frac{1}{b+l-1}\binom{a+1}{b+l-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \frac{l}{b}\binom{a+l-1}{b-1} & \frac{l-1}{b+1}\binom{a+l-1}{b} & \frac{l-2}{b+2}\binom{a+l-1}{b+1} & \cdots & \frac{1}{b+l-1}\binom{a+l-1}{b+l-2} & 0 \\ \binom{a+l}{b} & \binom{a+l}{b+1} & \binom{a+l}{b+2} & \cdots & \binom{a+l-1}{b+l-1} & \binom{a+l}{b+l} \end{pmatrix}$$

We get the following determinant

$$\frac{l!}{b(b+1)(b+2)\cdots(b+l-2)} \binom{a+l}{b+l} |N|,$$

where N is the matrix

$$(6.1.5) \qquad \begin{pmatrix} \binom{a}{b-1} & \binom{a}{b} & \binom{a}{b+1} & \cdots & \binom{a}{b+l-2} \\ \binom{a+1}{b-1} & \binom{a+1}{b} & \binom{a+1}{b+1} & \cdots & \binom{a+1}{b+l-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{a+l-1}{b-1} & \binom{a+l-1}{b} & \binom{a+l-1}{b+1} & \cdots & \binom{a+l-1}{b+l-2} \end{pmatrix}$$

By induction we get the following formula:

$$\frac{l!}{b(b+1)(b+2)\cdots(b+l-2)} \binom{a+l}{b+l} \frac{\binom{a}{b-1}\cdots\binom{a+l-1}{b+l-2}}{\binom{a-b+2}{1}\binom{a-b+3}{2}\cdots\binom{a-b+l}{l-1}},$$

and the formula 6.1.2 follows. $\qquad \square$

Lemma 6.2. *Let a be an integer with $a \geq 0$. Then the following is true: The determinant of the matrix*

$$(6.2.1) \qquad P = \begin{pmatrix} \binom{a}{a} & \binom{a-1}{a-1} & \binom{a-2}{a-2} & \cdots & \binom{a-l}{a-l} \\ \binom{a}{a-1} & \binom{a-1}{a-2} & \binom{a-2}{a-3} & \cdots & \binom{a-l}{a-l-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{a}{a-l+1} & \binom{a-1}{a-l} & \binom{a-2}{a-l-1} & \cdots & \binom{a-l}{a-2l+1} \\ \binom{a}{a-l} & \binom{a-1}{a-l-1} & \binom{a-2}{a-l-2} & \cdots & \binom{a-l}{a-2l} \end{pmatrix}$$

equals

$$(6.2.2) \qquad\qquad\qquad\qquad |P| = (-1)^{l+1}.$$

Proof. The matrix P equals the matrix

$$(6.2.3) \qquad P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-l \\ \frac{a(a-1)}{2!} & \frac{(a-1)(a-2)}{2!} & \frac{(a-2)(a-3)}{2!} & \cdots & \frac{(a-l)(a-l-1)}{2!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{a}{a-l} & \binom{a-1}{a-l-1} & \binom{a-2}{a-l-2} & \cdots & \binom{a-l}{a-2l} \end{pmatrix}.$$

We get the Vandermonde-matrix

$$(6.2.4) \qquad \frac{1}{1!2!\cdots l!} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a & a-1 & a-2 & \cdots & a-l \\ a^2 & (a-1)^2 & (a-2)^2 & \cdots & (a-l)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a^l & (a-1)^l & (a-2)^l & \cdots & (a-l)^l \end{pmatrix},$$

which is easily seen to have determinant $(-1)^{l+1}$, and the lemma follows. $\qquad\square$

REFERENCES

[1] A. Grothendieck, Éléments de géométrie algébrique IV$_4$, Étude locale des schémas et des morphismes de schémas , *Publ. Math. IHES* 32 (1967)

[2] A. Grothendieck, Sur la classification des fibrés holomorphes sur la sphère de Riemann, *Amer. J. Math.* 79 (1957)

[3] G. Harder, Halbeinfache Gruppenscemata über vollständigen Kurven, *Invent. Math.* 6 (1968)

[4] H. Maakestad, Modules of principal parts on the projective line, *Arkiv för Matematik* 42 (2004) no. 2

[5] H. Maakestad, A note on principal parts on projective space and linear representations,*Proc. Amer. Math. Soc.* 133 (2005) no. 2

[6] P. Jahnke, I. Radloff, Splitting jet sequences, *preprint math.AG/0210454*

[7] D. Laksov, A. Thorup, Weierstrass points on schemes, *J. Reine Angew. Math.* no. 460 (1995)

[8] D. Perkinson, Curves in grassmannians, *Trans. AMS* no. 347 (1995)

[9] R. Piene, G. Sacchiero, Duality for rational normal scrolls, *Comm. Algebra* 12 (1984)

[10] R. Piene, Numerical characters of a curve in projective n-space, *Real and complex singularities* Proc. Ninth Nordic Summer School Oslo (1976)

[11] S. di Rocco, A. J. Sommese, Line bundles for which a projectivized jet bundle is a product, *Proc. Amer. Math. Soc.* 129, (2001)

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